CHAPTER 13

PROBABILISTIC RISK ANALYSIS

RANDOM VARIABLES

- Factors having probabilistic outcomes
- The probability that a cost, revenue, useful life, or other economic factor value will occur, is usually considered to be the subjectively estimated likelihood that an event (value) occurs
- Random variable information that is particularly helpful in decision making are the expected values and variances
- These values make the uncertainty associated with each alternative more explicit

RANDOM VARIABLES

- Capital letters such as X, Y, and Z are used to represent random variables
- Lower-case letters (x,y,z) denote the particular values that these variables take on in the sample space (I.e., the set of possible outcomes for each variable)

RANDOM VARIABLES

- When random variable X follows some *discrete* probability distribution, its mass function is usually indicated by *p*(x) and its cumulative distribution function by P(x)
- When X follows a *continuous* probability distribution, its probability density function function and it cumulative distribution function are usually indicated by *f*(x) and F(x), respectively

DISCRETE RANDOM VARIABLES

- A random variable X is discrete if it can take on a finite number of values ($x_1, x_2 \ldots x_L$)
- The probability that a discrete random variable X takes on the value x_i is given by

  $\Pr\{X = x_i\} = p(x_i)$ for $i = 1,2,\ldots,L$ (i is a *sequential index* of the discrete values, x_i, that the variable takes on)

  where $p(x_i) \geq 0$ and $\Sigma_i\ p(x_i) = 1$

CONTINUOUS RANDOM VARIABLES

A random variable is continuous if:

$$\Pr\{c < X < d\} = \int_c^d f(x)dx$$

In the nonnegative function f(x),this is the probability that X is within the set of real numbers (c,d)

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

The probability that the value X is less than or equal x = k, the cumulative distribution function F(x) for a continuous case is

$$\Pr\{X < k\} = F(k) = \int_{-\infty}^{k} f(x)dx$$

$$\Pr\{c < X < d\} = \int_c^d f(x)dx = F(d) - F(c)$$

- **In most applications, continuous random variables represent measured data, such as time, cost and revenue on a continuous scale**

MATHEMATICAL EXPECTATIONS AND SELECTED STATISTICAL MOMENTS

- The expected value of a single random variable X, (E(X), is a weighted average of the distributed values x that it takes on and is a measure of the central location of the distribution
- E(X) is the first moment of the random variable about the origin and is called the mean of the distribution

  $E(X) = \Sigma_i \, x_i \, p(\, x_i \,)$ for x discrete and $i = 1,2,\ldots,L$

  $E(X) = \int_{-\infty}^{\infty} [x - E(X)]^2 \, f(x) dx$ for x continuous

MATHEMATICAL EXPECTATIONS AND SELECTED STATISTICAL MOMENTS

- From binomial expansion of $[X - E(X)]^2$

$$V(X) = E(X^2) - [E(X)]^2$$

- V(X) is the second moment of the random variable around the origin : the expected value of X^2, minus the square of its mean
- V(X) is the variance of the random variable X

$$V(X) = \Sigma_i \, x^2 p(x_i) - [E(X)]^2 \text{ for x discrete}$$

$$V(X) = \int_{-\infty}^{\infty} x_i^2(x)dx - [E(X)]^2 \text{ for x continuous}$$

- The standard deviation of a random variable, SD(X) is the positive square root of the variance

$$SD(X) = [V(X)]^{1/2}$$

MULTIPLICATION OF A RANDOM VARIABLE BY A CONSTANT

- When a random variable, X, is multiplied by a constant, c, the expected value E(cX), and the variance, V(cX) are:

$$E(cX) = cE(X) = \Sigma_i \, cx_i \, p(x_i) \text{ for discrete}$$

$$E(cX) = cE(X) = \int_{-\infty}^{\infty} cx \, f(x) dx \text{ for continuous}$$

$$V(cX) = E\{ [cX - E(cX)]^2 \}$$

$$= E\{c^2X^2 - 2c^2X \cdot E(X) + c^2 [E(X)]^2 \}$$

$$= c^2 E\{ [X - E(X)]^2 \}$$

MULTIPLICATION OF TWO INDEPENDENT VARIABLES

- When a random variable, Z, is a product of two independent random variables, X and Y, the expected value, E(Z), and the variance, V(Z) are

$$Z = XY$$

$$E(Z) = E(X)\,E(Y)$$

$$V(Z) = E\,[XY - E(X)]^2$$

$$= E\,\{\, X^2Y^2 - 2XY\,E(XY) + [E(XY)]^2 \,\}$$

$$= EX^2\,EY^2 - [E(X)\,E(Y)]^2$$

But the variance of any random variable, V(RV), is

$$V(RV) = E[(RV)^2] - [E(RV)]^2$$

$$E[(RV)^2] = V(RV) + [E(RV)]^2$$

MULTIPLICATION OF TWO INDEPENDENT VARIABLES

$$V(Z) = \{ V(X) + [E(X)]^2 \} \{ V(Y) + [E(Y)]^2 \} - [E(X)]^2 [E(Y)]^2$$

Or

$$V(Z) = V(X) [E(Y)]^2 + V(Y) [E(X)]^2 + V(X) V(Y)$$

EVALUATION OF PROJECTS WITH DISCRETE RANDOM VARIABLES

- Expected value and variance concepts apply theoretically to long-run conditions in which it is assumed that the event is going to occur repeatedly
- However, application of these concepts is often useful when investments are not going to be made repeatedly over the long run

EVALUATION OF PROJECTS WITH CONTINUOUS RANDOM VARIABLES

Two Frequently Used Assumptions

- Uncertain cash-flow amounts are distributed according to the normal distribution
- Uncertain cash flow amounts are statistically independent
  - no correlation between cash flow amounts is assumed

EVALUATION OF PROJECTS WITH CONTINUOUS RANDOM VARIABLES

If there is a linear combination of two or more independent cash flow amounts (i.e., $PW = c_0F_0 + \ldots + c_NF_N$, where c_k values are coefficients and F_k values are periodic net cash flows) the expression V(PW) reduces to

$$V(PW) = \Sigma_{k=0}^{N} c_k^2 V(F_k)$$

$$E(PW) = \Sigma_{k=0}^{N} c_k E(F_k)$$

EVALUATION OF UNCERTAINTY USING MONTE CARLO SIMULATION

- Computer-assisted simulation tool for analyzing more complex project uncertainties
- Monte Carlo simulation generates random outcomes for probabilistic factors which imitate the randomness inherent in the original problem

EVALUATION OF UNCERTAINTY USING MONTE CARLO SIMULATION

- Construct an analytical model that represents the actual decision situation
- Develop a probability distribution from subjective or historical data for each uncertain factor in the model
- Sample outcomes are randomly generated by using probability distribution for each uncertain quantity and then used to determine a trial outcome for the model
- Repeating sampling process many times leads to a frequency distribution of trial outcomes, which are used to make probabilistic statements

DECISION TREES

- Also called decision flow networks and decision diagrams
- Powerful means of depicting and facilitating analysis of important problems, especially those that involve sequential decisions and variable outcomes over time
- Practical tool because it permits large complicated problems to be reduced to a series of smaller simple problems
- Enable objective analysis and decision making that includes explicit consideration of the risk and effect of the future

GENERAL PRINCIPLE OF DIAGRAMING

The Decision Tree Diagram Should Show the Following

(With square symbol to depict decision node and circle symbol to depict chance outcome node):

1. **All initial or immediate alternatives among which the decision maker wishes to choose**
2. **All uncertain outcomes and future alternatives the decision maker wishes to consider**

Note alternatives at any point and outcomes at any chance outcome node must be:

- **Mutually exclusive**
- **Collectively exhaustive; that is, one event must be chosen or something must occur if the decision point or outcome node is reached**